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Supersymmetry, the Barducci–Casalbuoni–Lusanna Lagrangian, and the Weyl group in d dimensions

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Abstract. We consider the Lie algebra of the d-dimensional Weyl group $\otimes$ U(1) which describes off-mass-shell particles in d-dimensional Minkowski space. We show that the algebra generated by the position R^μ, momentum P^μ, spin $W^{\mu\nu}$, and mass M, is invariant under a transformation, involving an arbitrary function of P^2, which is an evolution of the relativistic system. We also show that, for $d=4$, the algebra is invariant under a transformation (rest-frame rotation) generated by $W^\mu/(P^2)^{1/2}$, where W^μ is the Pauli–Lubanski spin pseudovector. We analyse the Lagrangian of a spinning point particle in d dimensions using the invariant relation $\pi_5 - \mathrm{i}\frac{1}{2}\xi_5 \approx 0$ which is a second-class constraint, and implies another second-class constraint $P\cdot\xi\approx 0$. We show that the Dirac brackets for position, momentum, and spin satisfy the Weyl group algebra, and that the infinitesimal supersymmetry transformation transverse to P^μ of this d-dimensional Lagrangian is of the same form as the aforementioned transformation generated by $W^\mu/(P^2)^{1/2}$ for $d=4$. We quantise the system and construct an explicit realisation of the operators which generate the Weyl group plus Clifford algebras.

1. Introduction

There has recently been much interest in 'pseudoclassical' systems, i.e. classical systems described by anticommuting (odd Grassmann) variables. In fact, Allcock (1975a, b) and Casalbuoni (1976a, b) have given a comprehensive treatment of such systems both with and without second-class constraints (Dirac 1958, 1964, Hanson *et al* 1976), and have shown that, on quantisation, the anticommuting variables become Fermi operators (see also Klauder 1960 and Martin 1959). The classical limit of Fermi quantum systems, both unconstrained and constrained, was also studied by Droz-Vincent (1966) and Franke and Kálnay (1970) respectively, though they did not consider Grassmann variables. Barducci *et al* (1977) have emphasised the point of view that Grassmann variables are the classical ($\hbar\to 0$) limit of quantum operators with a bounded spectrum, since any odd Grassmann variable, or product of odd Grassmann variables, has zero square. We should like to give a purely classical example of how Grassmann variables can describe the physical properties of a system. Consider a classical nonrelativistic spherical top (spinning billiard ball) of radius a and mass M, in three space dimensions. If the angular momentum tensor relative to the centre of mass is S_{ij}, then the top has spin $S=\frac{1}{2}S_{ij}S_{ij}$, and its moment of inertia about its axis of rotation is $I=2Ma^2/5$. In order to obtain a point particle of mass M and spin S, one might naively think that we simply take the limit $a\to 0$ keeping M and S constant. However, this limit is unphysical since the rotational energy $S^2/2I$ blows up as $1/a^2$. To prevent this, S must go to zero at least as rapidly as a (physically, $S \sim a$ corresponds to a point on the surface of the sphere moving with constant speed as $a \to 0$). Mathematically, the way to keep the correct algebraic properties of S_{ij} and yet to have $S^2 = 0$ is to write $S_{ij} = -\mathrm{i}\frac{1}{2}[\xi_i, \xi_j]$ where ξ_i are odd Grassmann variables. In the Lagrangian (Casalbuoni 1976a, Berezin and Marinov 1977, Casalbuoni 1976c)

$$L = \mathrm{i}\tfrac{1}{2}\boldsymbol{\xi}(t)\cdot\dot{\boldsymbol{\xi}}(t) + \tfrac{1}{2}m\dot{\boldsymbol{x}}^2(t) \tag{1.1}$$

describing a free spinning nonrelativistic particle, the odd Grassmann variables ξ_i satisfy the Dirac brackets $\{\xi_i, \xi_j\}^* = -\mathrm{i}\delta_{ij}$, thereby giving the correct angular momentum Dirac brackets for S_{ij}.

The intrinsic spin of a relativistic particle can also be described by odd Grassmann variables, in fact it was in the context of the spinning relativistic string (Neveu and Schwarz 1971, Ramond 1971) that they, and the related supersymmetry†, first made their appearance. Barducci *et al* (1976) (BCL) have studied the Lagrangian

$$L = -\mathrm{i}\tfrac{1}{2}\xi_5(\tau)\dot{\xi}_5(\tau) - \mathrm{i}\tfrac{1}{2}\xi(\tau)\cdot\dot{\xi}(\tau) - mc[(\dot{x}(\tau) - \mathrm{i}\xi(\tau)\dot{\xi}_5(\tau)/mc)^2]^{1/2} \tag{1.2}$$

where $x^\mu(\tau)$ is a c-number vector, $\xi^\mu(\tau)$ is an odd Grassmann pseudovector, and $\xi_5(\tau)$ is an odd Grassmann pseudoscalar in four-dimensional Minkowski space (our metric convention is $g_{00} = 1$, $g_{ij} = -\delta_{ij}$, $g_{0i} = 0$), and showed that it described a pseudoclassical Dirac particle. A related Lagrangian involving Lagrange multipliers was studied by Berezin and Marinov (1977) who reached the same conclusion. The Lagrangian equation (1.2) was derived by Brink *et al* (1977) by eliminating the 'one-dimensional supergravity' fields $e(\tau)$ and $\chi(\tau)$ from the Lagrangian (Brink *et al* 1976, Collins and Tucker 1977)

$$L = \tfrac{1}{2}[\dot{x}^2/e + em^2c^2 - \mathrm{i}(\xi\cdot\dot{\xi} + \xi_5\dot{\xi}_5) - \mathrm{i}\chi(\xi\cdot\dot{x}/e - mc\xi_5)] \tag{1.3}$$

using the Euler–Lagrange equations for $e(\tau)$ and $\xi_5(\tau)$. The invariance (up to a τ-derivative) of the Lagrangian, equation (1.3), under τ-dependent supersymmetry transformations (Brink *et al* 1976) carries through to the Lagrangian equation (1.2) which changes by a τ-derivative under (Barducci *et al* 1976, Brink *et al* 1977)

$$\begin{aligned} &\xi^\mu(\tau) \to \xi^\mu(\tau) + \varepsilon_5(\tau)P^\mu(\tau)/mc \\ &\xi_5(\tau) \to \xi_5(\tau) + \varepsilon_5(\tau) \\ &x^\mu(\tau) \to x^\mu(\tau) - \mathrm{i}\varepsilon_5(\tau)\xi^\mu(\tau)/mc \end{aligned} \tag{1.4}$$

where $P^\mu = mc(\dot{x}^\mu - \mathrm{i}\xi^\mu\dot{\xi}_5/mc)/[(\dot{x} - \mathrm{i}\xi\dot{\xi}_5/mc)^2]^{1/2}$ and $\varepsilon_5(\tau)$ is an infinitesimal pseudoscalar odd Grassmann variable. Di Vecchia and Ravndal (1979) have recently studied a much simpler Lagrangian which is, however, not invariant under general τ-reparameterisations. The ideas which we develop in this paper could be equally well applied to that Lagrangian.

In this paper, we shall study the Lagrangian, equation (1.2), in d-dimensional Minkowski space, constructing the fully constrained Dirac brackets and quantising the system. The layout of the paper is as follows. In § 2, we treat the Weyl group, the group of Poincaré transformations plus dilatations in d-dimensional Minkowski space, which describes off-mass-shell relativistic particles (Almond 1973, 1974). We construct the position operator R^μ, and the spin operator $W^{\mu\nu}$ from the Hilbert space generators of

† There are many reviews of this topic available (see e.g. Corwin *et al* 1975, Ogievetskii and Mezincescu 1975, Ferrara 1976, Fayet and Ferrara 1977).

the Weyl group in quantum mechanics, and give the commutators which they, the momentum P^μ, and the mass M satisfy. We show that this algebra is invariant under a transformation involving an arbitrary function of P^2, this transformation corresponding to an evolution of the physical system. We also show that the algebra is invariant under rest-frame rotations, which, for $d=4$, are generated by $W^\mu/(P^2)^{1/2}$ where W^μ is the Pauli–Lubanski spin pseudovector. We mention that an infinitesimal rest-frame rotation for $d=4$ is of the same form as an infinitesimal supersymmetry transformation transverse to P^μ, which is discussed in § 3. We suggest that this phenomenon, which occurs also in the nonrelativistic case, is connected with the existence of a set of matrices which is a representation of both the Clifford algebra C_3 and the rotation group Lie algebra SO(3).

In § 3, we give a classical treatment of the BCL Lagrangian in d dimensions which differs from that of BCL in that our 'invariant relation' (which is required to give a physical solution) is of the form† $\pi_5(0)-\mathrm{i}\frac{1}{2}\xi_5(0)\approx 0$, which is a constant of the motion and a second-class constraint, and which gives another second-class constraint $P\cdot\xi\approx 0$. The Dirac brackets which we finally obtain involving the position, momentum, and spin are just those of the Weyl group given in § 2. In fact, these second-class constraints have already been used by Bachas (1978) to obtain the Dirac brackets for the system for $d=4$. However, since he also applied the mass-shell constraint and the gauge-fixing constraint $x^0-c\tau\approx 0$, his results are different from ours. We are primarily concerned with the *off-mass-shell* Dirac brackets and their connection with the Weyl group. We also calculate the Dirac brackets involving the generators of supersymmetry transformations along P^μ, and transverse to P^μ. The latter has the same Dirac brackets as those of $W^\mu/(P^2)^{1/2}$ for $d=4$ calculated in § 2.

In § 4, we quantise the system, giving an explicit realisation of the position, momentum, spin, and supersymmetry generators in a p^μ-basis involving γ-matrices.

2. The Weyl group in d dimensions

The group of Poincaré transformations and dilatations on Minkowski space–time has been treated in detail for $d=4$ (Almond 1973) (for a mini-review see Almond (1974)) so we shall be very brief. We are actually interested in the direct sum of the Weyl algebra with U(1):

$$\begin{aligned}
&[M^{\mu\nu}, M^{\rho\sigma}] = \mathrm{i}\hbar(M^{\mu\rho}g^{\nu\sigma} - M^{\mu\sigma}g^{\nu\rho} + M^{\nu\sigma}g^{\mu\rho} - M^{\nu\rho}g^{\mu\sigma})\\
&[M^{\mu\nu}, P^\sigma] = -\mathrm{i}\hbar(P^\mu g^{\nu\sigma} - P^\nu g^{\mu\sigma})\\
&[P^\mu, P^\nu] = 0, \qquad [D, M^{\mu\nu}] = 0, \qquad [D, P^\mu] = -\mathrm{i}\hbar P^\mu\\
&[M^{\mu\nu}, M] = 0, \qquad [P^\mu, M] = 0, \qquad [D, M] = 0
\end{aligned} \tag{2.1}$$

where $M^{\mu\nu}$, P, D, and M are the Hermitian operators generating Lorentz transformations, translations, dilatations, and U(1) transformations respectively in Hilbert space. We now define the position operator

$$R^\mu = \tfrac{1}{2}[P^\mu/P^2, D]_+ - [P_\nu, M^{\mu\nu}]_+/2P^2 \tag{2.2}$$

† The symbol ≈ 0 means weakly equal to zero in the sense of Dirac (1958, 1964), i.e. its Poisson bracket with any other dynamical variable is not necessarily zero.

(where $[A, B]_+ \equiv AB + BA$) and the spin operator (Nyborg 1964, Kolsrud 1967)

$$W^{\mu\nu} = (g^{\mu\rho} - P^\mu P^\rho/P^2)(g^{\nu\sigma} - P^\nu P^\sigma/P^2)M_{\rho\sigma}. \tag{2.3}$$

(The Hermiticity of this expression for $W^{\mu\nu}$ may be checked using the second of equations (2.1), which gives $P_\rho P_\sigma M^{\rho\sigma} = 0$ and $P_\rho M^{\rho\sigma} P_\sigma = -i\hbar P^2(d-1)$.) The operators R^μ and $W^{\mu\nu}$ satisfy

$$\begin{aligned}
&[M^{\mu\nu}, R^\sigma] = -i\hbar(R^\mu g^{\nu\sigma} - R^\nu g^{\mu\sigma}), \qquad [R^\mu, P^\nu] = -i\hbar g^{\mu\nu} \\
&[D, R^\mu] = i\hbar R^\mu, \qquad [M, R^\mu] = 0, \qquad [R^\mu, R^\nu] = -i\hbar W^{\mu\nu}/P^2 \\
&[M^{\mu\nu}, W^{\rho\sigma}] = i\hbar(W^{\mu\rho} g^{\nu\sigma} - W^{\mu\sigma} g^{\nu\rho} + W^{\nu\sigma} g^{\mu\rho} - W^{\nu\rho} g^{\mu\sigma}) \\
&[P^\mu, W^{\rho\sigma}] = 0, \qquad [D, W^{\rho\sigma}] = 0, \qquad [M, W^{\rho\sigma}] = 0 \\
&[W^{\mu\nu}, W^{\rho\sigma}] = i\hbar\left(W^{\mu\rho}\left(g^{\nu\sigma} - \frac{P^\nu P^\sigma}{P^2}\right) - W^{\mu\sigma}\left(g^{\nu\rho} - \frac{P^\nu P^\rho}{P^2}\right)\right. \\
&\qquad\qquad \left. + W^{\nu\sigma}\left(g^{\mu\rho} - \frac{P^\mu P^\rho}{P^2}\right) - W^{\nu\rho}\left(g^{\mu\sigma} - \frac{P^\mu P^\sigma}{P^2}\right)\right) \\
&[R^\mu, W^{\rho\sigma}] = -i\hbar(P^\rho W^{\sigma\mu} - P^\sigma W^{\rho\mu})/P^2
\end{aligned} \tag{2.4}$$

and the algebra, equations (2.1) and (2.4), which describes an off-mass-shell relativistic particle, has as invariants for $d = 4$, the operators $M, \frac{1}{2}W^{\mu\nu}W_{\mu\nu}$, and sign (P^2) which give the on-mass-shell mass, spin, and sign of the momentum squared of the particle. For $d > 4$, there are other spin-type invariants e.g. for $d = 5$, $\frac{1}{4}\varepsilon^{\mu\nu\rho\sigma\tau} P_\mu W_{\nu\rho} W_{\sigma\tau}/(P^2)^{1/2}$ is also an invariant. The classical versions of the above equations are obtained by the substitutions $\frac{1}{2}[A, B]_+ \to AB$, and $[A, B]/i\hbar \to \{A, B\}$, the Poisson bracket. Note that the commutators/brackets of equations (2.1) and (2.4) are fundamental equations which are valid for *any* off-mass-shell relativistic system. They have been found by explicit calculation for the spherical top (for $d = 4$) by Hanson and Regge (1974), the G_4 supersymmetric model (for $d = 4$) by Casalbuoni (1976a), and for the massless relativistic string (for general d) by Almond (1978). The particle is put onto the mass-shell classically by applying to the Poisson brackets of equation (2.4) the mass-shell constraint together with a 'gauge-fixing' constraint which eliminates one component of R^μ (usually R^0, R^+, or D)

$$\begin{aligned}
&P^2 - M^2c^2 \approx 0 \\
&R^0 \approx \alpha \quad \text{or } R^+ \approx \beta \quad \text{or } D \approx \gamma
\end{aligned} \tag{2.5}$$

with α, β, γ constant.

In quantum mechanics, we also have a unitary parity operator $\mathscr{P}$, and an antiunitary time-reversal operator $\mathscr{T}$ which satisfy (Almond 1973)

$$\begin{aligned}
&\mathscr{P}M^{\mu\nu}\mathscr{P}^{-1} = \eta(\mu)\eta(\nu)M^{\mu\nu} && \mathscr{T}M^{\mu\nu}\mathscr{T}^{-1} = -\eta(\mu)\eta(\nu)M^{\mu\nu} \\
&\mathscr{P}P^\mu\mathscr{P}^{-1} = \eta(\mu)P^\mu && \mathscr{T}P^\mu\mathscr{T}^{-1} = \eta(\mu)P^\mu \\
&\mathscr{P}D\mathscr{P}^{-1} = D && \mathscr{T}D\mathscr{T}^{-1} = -D \\
&\mathscr{P}R^\mu\mathscr{P}^{-1} = \eta(\mu)R^\mu && \mathscr{T}R^\mu\mathscr{T}^{-1} = -\eta(\mu)R^\mu \\
&\mathscr{P}W^{\mu\nu}\mathscr{P}^{-1} = \eta(\mu)\eta(\nu)W^{\mu\nu} && \mathscr{T}W^{\mu\nu}\mathscr{T}^{-1} = -\eta(\mu)\eta(\nu)W^{\mu\nu} \\
&\mathscr{P}M\mathscr{P}^{-1} = M && \mathscr{T}M\mathscr{T}^{-1} = M
\end{aligned} \tag{2.6}$$

where $\eta(0) = +1$, $\eta(i) = -1$.

An interesting fact about equations (2.1), (2.4), and (2.6) is that they are invariant under the transformation

$$
\begin{aligned}
&D \equiv D(0) \to D(\tau) = U[f] D U^{-1}[f] = D(0) + f(P^2, \tau) \\
&R^\mu \equiv R^\mu(0) \to R^\mu(\tau) = U[f] R^\mu U^{-1}[f] = R^\mu(0) + f(P^2, \tau) P^\mu / P^2 \qquad (2.7)\\
&\mathcal{T} \equiv \mathcal{T}(0) \to \mathcal{T}(\tau) = U[f] \mathcal{T} U^{-1}[f] = \exp\left(-\frac{\mathrm{i}}{\hbar} \int_{M^2c^2}^{P^2} \frac{f(\sigma, \tau)}{\sigma} \, \mathrm{d}\sigma\right) \mathcal{T}(0)
\end{aligned}
$$

where $U[f] = \exp[-(\mathrm{i}/2\hbar) \int_{M^2c^2}^{P^2} f(\sigma, \tau) \, \mathrm{d}\sigma/\sigma]$, and $f(P^2, \tau)$ is an arbitrary real function of P^2 and the evolution parameter τ. (In checking this invariance, remember that $[D, F(P^2)] = -2\mathrm{i}\hbar P^2 \, \mathrm{d}F/\mathrm{d}P^2$ for any function $F(P^2)$.) Equation (2.7) describes an evolution of the off-mass-shell physical system (that is why we have to redefine the time-reversal operator too). The arbitrary function $f(P^2, \tau)$ is just that which appears in the invariance of the action integral $S = \int_{\tau_2}^{\tau_1} L \, \mathrm{d}\tau$ (where L is a Lagrangian describing a free relativistic particle of mass m) under the transformation $\tau \to f(m^2c^2, \tau)$.

Rest-frame rotations are generated by the spin tensor $W^{\mu\nu}$ of equation (2.3). However, for $d = 4$, something special happens. In that case we can define† the Pauli–Lubanski spin pseudovector $W^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} M_{\nu\rho} P_\sigma = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} W_{\nu\rho} P_\sigma$, and even more interesting is the operator $W^\mu/(P^2)^{1/2}$, which has the commutators (Almond 1973):

$$
\left.
\begin{aligned}
&\left[M^{\mu\nu}, \frac{W^\sigma}{(P^2)^{1/2}}\right] = -\mathrm{i}\hbar\left(\frac{W^\mu}{(P^2)^{1/2}} g^{\nu\sigma} - \frac{W^\nu}{(P^2)^{1/2}} g^{\mu\sigma}\right) \\
&\left[P^\mu, \frac{W^\nu}{(P^2)^{1/2}}\right] = 0, \qquad \left[D, \frac{W^\mu}{(P^2)^{1/2}}\right] = 0, \qquad \left[M, \frac{W^\mu}{(P^2)^{1/2}}\right] = 0 \\
&\left[W^{\mu\nu}, \frac{W^\sigma}{(P^2)^{1/2}}\right] = -\mathrm{i}\hbar\left(\frac{W^\mu}{(P^2)^{1/2}}\left(g^{\nu\sigma} - \frac{P^\nu P^\sigma}{P^2}\right) - \frac{W^\nu}{(P^2)^{1/2}}\left(g^{\mu\sigma} - \frac{P^\mu P^\sigma}{P^2}\right)\right) \\
&\left[R^\mu, \frac{W^\nu}{(P^2)^{1/2}}\right] = \frac{\mathrm{i}\hbar P^\nu W^\mu}{(P^2)^{3/2}}
\end{aligned}
\right\} \qquad (2.8a)
$$

$$
\left[\frac{W^\mu}{(P^2)^{1/2}}, \frac{W^\nu}{(P^2)^{1/2}}\right] = -\mathrm{i}\hbar W^{\mu\nu} \qquad (2.8b)
$$

with similar classical expressions in terms of Poisson brackets. The remarkable fact is that equations (2.8a) are identical to the Dirac brackets of Ξ^μ with the various physical quantities in § 3 (see equation (3.28)) and equations (2.8) are identical to the commutators of $\sqrt{\frac{1}{2}\hbar}\,\Xi^\mu$ with the various physical quantities in § 4, where $-\mathrm{i}\Xi^\mu$ is the generator of supersymmetry transformations transverse to P^μ, and is classically an odd Grassmann variable. In fact equations (2.1), (2.4), and (2.6) are invariant (for $d = 4$) under the transformation (infinitesimal rest-frame rotation):

$$
\begin{aligned}
&R^\mu \to R^\mu + \hbar\varepsilon \cdot P W^\mu/(P^2)^{1/2} \\
&W^{\mu\nu} \to W^{\mu\nu} + \hbar\left(\varepsilon^\mu - \frac{\varepsilon \cdot P P^\mu}{P^2}\right)\frac{W^\nu}{(P^2)^{1/2}} - \hbar\left(\varepsilon^\nu - \frac{\varepsilon \cdot P P^\nu}{P^2}\right)\frac{W^\mu}{(P^2)^{1/2}} \qquad (2.9a)\\
&M^{\mu\nu} \to M^{\mu\nu} + \hbar\varepsilon^\mu \frac{W^\nu}{(P^2)^{1/2}} - \hbar\varepsilon^\nu \frac{W^\mu}{(P^2)^{1/2}}
\end{aligned}
$$

† Our convention is $\varepsilon^{0123} = +1$.

$$\mathscr{P} \to (1+2\mathrm{i}\varepsilon^0 W^0/(P^2)^{1/2})\mathscr{P}$$
$$\mathscr{T} \to (1+2\mathrm{i}\varepsilon^0 W^0/(P^2)^{1/2}\mathscr{T} \tag{2.9b}$$

where ε^μ is the infinitesimal parameter of the transformation. (The invariance of equations (2.6) involving $M^{\mu\nu}$ may be checked component by component, e.g. we find

$$\left(1+\frac{2\mathrm{i}\varepsilon^0 W^0}{(P^2)^{1/2}}\right)\mathscr{P}\left[M^{ij}+\hbar\left(\frac{\varepsilon^i W^j}{(P^2)^{1/2}}-\frac{\varepsilon^j W^i}{(P^2)^{1/2}}\right)\right]\mathscr{P}^{-1}\left(1-\frac{2\mathrm{i}\varepsilon^0 W^0}{(P^2)^{1/2}}\right)$$
$$=\eta(i)\eta(j)\left[M^{ij}+\hbar\left(\frac{\varepsilon^i W^j}{(P^2)^{1/2}}-\frac{\varepsilon^j W^i}{(P^2)^{1/2}}\right)\right] \tag{2.10}$$

etc). Equations (2.9a) have exactly the same form as the infinitesimal supersymmetry transformation generated by $-\mathrm{i}\Xi^\mu$ for general d, which leaves invariant the Dirac brackets in § 3.

In attempting to understand this phenomenon, it is worth pointing out that the same thing occurs in nonrelativistic physics. The spin tensor S_{ij} (the analogue of $W^{\mu\nu}$) can be defined for general space dimension, and it has the commutators/Poisson brackets (the analogues of equations (2.1) and (2.4)):

$$[J_{ij}, J_{kl}] = -\mathrm{i}\hbar(J_{ik}\delta_{jl}-J_{il}\delta_{jk}+J_{jl}\delta_{ik}-J_{jk}\delta_{il})$$
$$[J_{ij}, S_{kl}] = -\mathrm{i}\hbar(S_{ik}\delta_{jl}-S_{il}\delta_{jk}+S_{jl}\delta_{ik}-S_{jk}\delta_{il}) \tag{2.11}$$
$$[S_{ij}, S_{kl}] = -\mathrm{i}\hbar(S_{ik}\delta_{jl}-S_{il}\delta_{jk}+S_{jl}\delta_{ik}-S_{jk}\delta_{il})$$

where $J_{ij} = -(R_iP_j - R_jP_i + S_{ij})$ is the angular momentum tensor. (There are, of course, other commutators involving T, R_i, P_i, and M, but since these quantities are unaffected by rest-frame rotations, they will not concern us here.) In three dimensions, we can also define the spin pseudovector $S_i = \frac{1}{2}\varepsilon_{ijk}S_{jk}$, which has the commutators:

$$[J_{ij}, S_k] = \mathrm{i}\hbar(S_i\delta_{jk}-S_j\delta_{ik})$$
$$[S_{ij}, S_k] = -\mathrm{i}\hbar(S_i\delta_{jk}-S_j\delta_{ik}) \tag{2.12a}$$
$$[S_i, S_j] = \mathrm{i}\hbar S_{ij} \tag{2.12b}$$

with similar expressions in terms of Poisson brackets. The generalisation of the Lagrangian equation (1.1) to an arbitrary number of space dimensions is straightforward and we find the Dirac brackets

$$\{J_{ij}, \xi_k\}^* = (\xi_i\delta_{jk}-\xi_j\delta_{ik})$$
$$\{S_{ij}, \xi_k\}^* = -(\xi_i\delta_{jk}-\xi_j\delta_{ik}) \tag{2.13}$$
$$\{\xi_i, \xi_j\}^* = -\mathrm{i}\delta_{ij}$$

where $S_{ij} = -\frac{1}{2}\mathrm{i}[\xi_i, \xi_j]$. On quantisation, equations (2.13) become

$$[J_{ij}, \xi_k] = \mathrm{i}\hbar(\xi_i\delta_{jk}-\xi_j\delta_{ik})$$
$$[S_{ij}, \xi_k] = -\mathrm{i}\hbar(\xi_i\delta_{jk}-\xi_j\delta_{ik}) \tag{2.14a}$$
$$[\xi_i, \xi_j]_+ = \hbar\delta_{ij}. \tag{2.14b}$$

We see that, classically, equations (2.12a) are of the same form as the first two of equations (2.13), and, quantum-mechanically, equations (2.12) are of the same form as equations (2.14a) and the definition of S_{ij} in terms of $(\frac{1}{2}\hbar)^{1/2}\xi_i$ (i.e. $\hbar S_{ij} = -\mathrm{i}[(\frac{1}{2}\hbar)^{1/2}\xi_i, (\frac{1}{2}\hbar)^{1/2}\xi_j]$). Furthermore, in three space dimensions, equations (2.11) are invariant under the rest-frame rotation

$$\begin{aligned} S_{ij} &\to S_{ij} - \hbar(\varepsilon_i S_j - \varepsilon_j S_i) \\ J_{ij} &\to J_{ij} + \hbar(\varepsilon_i S_j - \varepsilon_j S_i) \end{aligned} \tag{2.15}$$

whilst the same Dirac brackets, as well as equations (2.13), are invariant under

$$\begin{aligned} \xi_i &\to \xi_i + \alpha_i \\ S_{ij} &\to S_{ij} - \mathrm{i}(\alpha_i\xi_j + \xi_i\alpha_j) \\ J_{ij} &\to J_{ij} + \mathrm{i}(\alpha_i\xi_j + \xi_i\alpha_j) \end{aligned} \tag{2.16}$$

where α_i is an infinitesimal odd Grassmann parameter. (Note, incidentally, that the infinitesimal rest-frame rotation generated by $\varepsilon_{ij}S_{ij}$ (ε_{ij} an infinitesimal antisymmetric c-number parameter) can be considered as a supersymmetry transformation with a ξ_i-dependent parameter since $\varepsilon_{ij}S_{ij} = (-\mathrm{i}\varepsilon_{ij}\xi_i)\xi_j$.) This connection between rest-frame rotations in three dimensions and supersymmetry transformations is surely connected with the fact that there exists a set of matrices which satisfy both the angular momentum commutation relations of SO(3)

$$[S_i, S_j] = \mathrm{i}\hbar\varepsilon_{ijk}S_k \tag{2.17}$$

and the Clifford algebra C_3 of equation (2.14b). They are, of course, the Pauli spin matrices σ_i ($i = 1, 2, 3$) satisfying

$$\sigma_i\sigma_j = \delta_{ij} + \mathrm{i}\varepsilon_{ijk}\sigma_k \tag{2.18}$$

so that (Casalbuoni 1976a, Berezin and Marinov 1977, Casalbuoni 1976c) $S_i = \frac{1}{2}\hbar\sigma_i$ and $\xi_i = \sqrt{\frac{1}{2}\hbar}\sigma_i$.

Similarly, in the relativistic case, there exists a set of matrices satisfying the commutation relations of transverse-to-P^μ SO(3)

$$\left[\frac{W^\mu}{(P^2)^{1/2}}, \frac{W^\nu}{(P^2)^{1/2}}\right] = -\mathrm{i}\hbar\varepsilon^{\mu\nu\rho\sigma}P_\rho\frac{W_\sigma}{(P^2)^{1/2}}\Big/(P^2)^{1/2} \tag{2.19}$$

and the transverse-to-P^μ Clifford algebra C_3

$$[\Xi^\mu, \Xi^\nu]_+ = -\hbar(g^{\mu\nu} - P^\mu P^\nu/P^2). \tag{2.20}$$

They are given by Γ^μ ($\mu = 0, 1, 2, 3$) (Fradkin and Good 1961, Kolsrud 1967)†

$$\Gamma^\mu = \gamma^\rho\gamma^5(g^\mu_\rho - \mathbf{P}_\rho\mathbf{P}^\mu/p^2) \tag{2.21}$$

where $\mathbf{P}^\mu$ is the *matrix* $(\gamma^0\boldsymbol{\gamma}\cdot\boldsymbol{p} + \gamma^0(p^2)^{1/2}, \boldsymbol{p})$ satisfying

$$\Gamma^\mu\Gamma^\nu = -(g^{\mu\nu} - \mathbf{P}^\mu\mathbf{P}^\nu/p^2) - \mathrm{i}\varepsilon^{\mu\nu\rho\sigma}\mathbf{P}_\rho\Gamma_\sigma/(p^2)^{1/2} \tag{2.22}$$

so that $W^\mu/(P^2)^{1/2} = \frac{1}{2}\hbar\Gamma^\mu$ and $\Xi^\mu = (\frac{1}{2}\hbar)^{1/2}\Gamma^\mu$. (Note that the representation of Ξ^μ given in § 4 does *not* satisfy equation (2.19) too for $d = 4$.)

† Our γ-matrix convention is that of Bjorken and Drell (1964).

3. BCL Lagrangian in d dimensions: classical theory

The Lagrangian is that of equation (1.2) repeated here for convenience:

$$L=-\mathrm{i}\tfrac{1}{2}\xi_5\dot{\xi}_5-\mathrm{i}\tfrac{1}{2}\xi\cdot\dot{\xi}-mc[(\dot{x}-\mathrm{i}\xi\dot{\xi}_5/mc)^2]^{1/2} \tag{3.1}$$

where $x^\mu(\tau)$ is a c-number vector, $\xi^\mu(\tau)$ is an odd Grassmann pseudovector, and $\xi_5(\tau)$ (the correct notation is $\xi_{d+1}(\tau)$ but no confusion should arise) is an odd Grassmann pseudoscalar in d-dimensional Minkowski space. The canonical momenta are

$$P^\mu(\tau)=-\frac{\partial L}{\partial\dot{x}_\mu}=\frac{mc(\dot{x}^\mu-\mathrm{i}\xi^\mu\dot{\xi}_5/mc)}{[(\dot{x}-\mathrm{i}\xi\dot{\xi}_5/mc)^2]^{1/2}} \tag{3.2a}$$

$$\pi^\mu(\tau)=\frac{\partial L}{\partial\dot{\xi}_\mu}=\tfrac{1}{2}\mathrm{i}\xi^\mu \tag{3.2b}$$

$$\pi_5(\tau)=\frac{\partial L}{\partial\dot{\xi}_5}=\tfrac{1}{2}\mathrm{i}\xi_5-\frac{\mathrm{i}P\cdot\xi}{mc} \tag{3.2c}$$

and the Euler–Lagrange equations are

$$\dot{P}^\mu(\tau)=0 \qquad (\text{i.e. } P^\mu(\tau)=P^\mu) \tag{3.3a}$$

$$\dot{\xi}^\mu(\tau)=P^\mu\dot{\xi}_5(\tau)/mc \qquad (\text{i.e. } \dot{\xi}^i(P)=L^{-1}(P)^i_{\ \mu}\dot{\xi}^\mu(\tau)=0)\dagger \tag{3.3b}$$

$$\dot{\xi}_5(\tau)=P\cdot\dot{\xi}(\tau)/mc. \tag{3.3c}$$

The canonical Poisson brackets are:

$$\{x^\mu,P^\nu\}=-g^{\mu\nu}, \qquad \{\pi^\mu,\xi^\nu\}=-g^{\mu\nu}, \qquad \{\pi_5,\xi_5\}=-1 \tag{3.4}$$

and the canonical momenta and coordinates are not independent, satisfying the constraints‡

$$\chi\equiv P^2-m^2c^2\approx 0 \tag{3.5a}$$

$$\chi_D\equiv\pi_5-\tfrac{1}{2}\mathrm{i}\xi_5+P\cdot\pi/mc+\mathrm{i}P\cdot\xi/2mc\approx 0 \tag{3.5b}$$

$$\chi^\mu\equiv\pi^\mu-\tfrac{1}{2}\mathrm{i}\xi^\mu\approx 0 \tag{3.5c}$$

where χ and χ_D are first-class satisfying

$$\begin{aligned}&\{\chi,\chi\}=0, \qquad \{\chi,\chi_D\}=0, \qquad \{\chi,\chi^\mu\}=0\\ &\{\chi_D,\chi_D\}=-\mathrm{i}\chi/m^2c^2, \qquad \{\chi_D,\chi^\mu\}=0\end{aligned} \tag{3.6}$$

whilst χ^μ is second-class satisfying

$$\{\chi^\mu,\chi^\nu\}=\mathrm{i}g^{\mu\nu}\equiv C^{\mu\nu}. \tag{3.7}$$

The generators of Lorentz transformations, dilatations, and pseudoscalar and pseudovector supersymmetry transformations are

$$M^{\mu\nu}(\tau)=P^\mu x^\nu(\tau)-P^\nu x^\mu(\tau)+\pi^\mu(\tau)\xi^\nu(\tau)-\pi^\nu(\tau)\xi^\mu(\tau) \tag{3.8a}$$

$$D(\tau)=P\cdot x(\tau)+\pi(\tau)\cdot\xi(\tau)+\pi_5(\tau)\xi_5(\tau) \tag{3.8b}$$

† $L^{-1}(P)^\lambda_{\ \mu}$ is the matrix which takes P^μ into $((P^2)^{1/2},\mathbf{0})$, see equation (4.2).

‡ Our definition of χ_D differs from that of BCL by a factor i/mc.

$$G_5(\tau) = -(\pi_5(\tau) + \mathrm{i}\tfrac{1}{2}\xi_5(\tau)) \tag{3.8c}$$

$$G^\mu(\tau) = -(\pi^\mu(\tau) + \mathrm{i}\tfrac{1}{2}\xi^\mu(\tau) - \mathrm{i}P^\mu\xi_5(\tau)/mc) \tag{3.8d}$$

the supersymmetry generators satisfying

$$\{G_5, G_5\} = -\mathrm{i}, \qquad \{G^\mu, G^\nu\} = -\mathrm{i}g^{\mu\nu}, \qquad \{G^\mu, G_5\} = \mathrm{i}P^\mu/mc. \tag{3.9}$$

We now define Dirac brackets $\{\ ,\ \}^*$ compatible with the second-class constraint equation (3.5c) by

$$\{A, B\}^* = \{A, B\} - \{A, \chi_\mu\}(C^{-1})^{\mu\nu}\{\chi_\nu, B\} \tag{3.10}$$

which tells us that we can put π^μ strongly equal to $\mathrm{i}\frac{1}{2}\xi^\mu$ everywhere if we use

$$\{\xi^\mu, \xi^\nu\}^* = \mathrm{i}g^{\mu\nu} \tag{3.11}$$

so we write ·

$$\left.\begin{aligned} M^{\mu\nu}(\tau) &= P^\mu x^\nu(\tau) - P^\nu x^\mu(\tau) + \mathrm{i}\tfrac{1}{2}[\xi^\mu(\tau), \xi^\nu(\tau)] \\ D(\tau) &= P\cdot x(\tau) + \pi_5(\tau)\xi_5(\tau) \\ G_5(\tau) &= -(\pi_5(\tau) + \mathrm{i}\tfrac{1}{2}\xi_5(\tau)) \\ G^\mu(\tau) &= -\mathrm{i}\xi^\mu(\tau) + \mathrm{i}P^\mu\xi_5(\tau)/mc \end{aligned}\right. \tag{3.12}$$

and the Dirac brackets $\{\ ,\ \}^*$ of the Gs are unchanged from the Poisson brackets equations (3.9).

So far our treatment is that of BCL. Since the canonical Hamiltonian vanishes, they show that the general Hamiltonian is a linear combination of the two first-class constraints χ and χ_D

$$H = \rho_1(P^2 - m^2c^2) + (\lambda_1\pi_5 + \lambda_2\xi_5 + \lambda_3 P\cdot\xi)(\pi_5 - \mathrm{i}\tfrac{1}{2}\xi_5 + \mathrm{i}P\cdot\xi/mc) \tag{3.13}$$

where ρ_1, λ_1, λ_2, and λ_3 are c-number constants. Equation (3.13) can be rewritten as

$$H = \rho_1(P^2 - m^2c^2) + (\lambda_1\pi_5 + \lambda_2\xi_5)\mathrm{i}P\cdot\xi/mc$$
$$+ \lambda_3 P\cdot\xi(\pi_5 - \mathrm{i}\tfrac{1}{2}\xi_5) - (\mathrm{i}\tfrac{1}{2}\lambda_1 + \lambda_2)\pi_5\xi_5. \tag{3.14}$$

Clearly, one choice of constants that will simplify the equations of motion is $\lambda_2 = -\mathrm{i}\frac{1}{2}\lambda_1$, when we find

$$H = \rho_1(P^2 - m^2c^2) + \rho_2(\pi_5 - \mathrm{i}\tfrac{1}{2}\xi_5)P\cdot\xi \tag{3.15}$$

and BCL actually made this choice for their quantum Hamiltonian. They also noted that in order to obtain a physical solution to the Hamiltonian equations of motion, an 'invariant relation' of the form $\pi_5(0) - \mu\xi_5(0) \approx 0$ is needed. The point about the Hamiltonian, equation (3.15), is that

$$(\mathrm{d}/\mathrm{d}\tau)(\pi_5(\tau) - \mu\xi_5(\tau)) = \{\pi_5(\tau) - \mu\xi_5(\tau), H\}^* = \rho_2(\mu + \tfrac{1}{2}\mathrm{i})P\cdot\xi \tag{3.16}$$

so that $\mathrm{d}(\pi_5(\tau) - \mu\xi_5(\tau))/\mathrm{d}\tau = 0$ for $\mu = -\frac{1}{2}\mathrm{i}$, and is ≈ 0 for $\mu = \frac{1}{2}\mathrm{i}$ because then $\chi_D \approx \mathrm{i}P\cdot\xi/mc$, so for these two values of μ, $\pi_5(\tau) - \mu\xi_5(\tau)$ is at least weakly conserved in τ, and therefore satisfies the condition for an invariant relation (Hanson *et al* 1976). BCL analysed the case $\mu = -\frac{1}{2}\mathrm{i}$ in which the second-class constraint $\pi_5 + \mathrm{i}\frac{1}{2}\xi_5 \approx 0$ converts χ_D into the Dirac equation constraint $\mathrm{i}\frac{1}{2}(P\cdot\xi/mc - \xi_5) \approx 0$ which is first-class and implies the mass-shell constraint $\chi \approx 0$ via its Dirac bracket with itself.

We shall now look at the case $\mu=\frac{1}{2}\mathrm{i}$. We now need the Dirac brackets compatible with the constraint

$$\chi_5' \equiv \pi_5 - \mathrm{i}\tfrac{1}{2}\xi_5 \approx 0 \tag{3.17}$$

which is second-class satisfying

$$\{\chi_5', \chi_5'\}^* = \mathrm{i}. \tag{3.18}$$

Furthermore, we have

$$\{\chi_5', \chi\}^* = 0, \qquad \{\chi_5', \chi_D\}^* = \mathrm{i} \tag{3.19}$$

so that although $\chi \approx 0$ is still first-class, $\chi_D \approx 0$ is now second-class. We need new Dirac brackets $\{\ ,\ \}^{**}$ defined by

$$\{A, B\}^{**} = \{A, B\}^* - \{A, \chi_5'\}^*(\mathrm{i})^{-1}\{\chi_5', B\}^* \tag{3.20}$$

which tells us that we can put π_5 strongly equal to $\mathrm{i}\frac{1}{2}\xi_5$ everywhere if we use

$$\{\xi_5, \xi_5\}^{**} = \mathrm{i}. \tag{3.21}$$

So we write

$$\begin{aligned} M^{\mu\nu}(\tau) &= P^\mu x^\nu(\tau) - P^\nu x^\mu(\tau) + \mathrm{i}\tfrac{1}{2}[\xi^\mu(\tau), \xi^\nu(\tau)] \\ D(\tau) &= P\cdot x(\tau) \\ G_5(\tau) &= 0 \\ G^\mu(\tau) &= -\mathrm{i}\xi^\mu(\tau) + \mathrm{i}P^\mu\xi_5(\tau)/mc. \end{aligned} \tag{3.22}$$

We now need to apply $P\cdot\xi \approx 0$ as a second-class constraint. It satisfies

$$\{P\cdot\xi, P\cdot\xi\}^{**} \approx \mathrm{i}P^2. \tag{3.23}$$

However, instead of defining new Dirac brackets, we find it more illuminating, as did Bachas (1978), to redefine the *physical quantities* by

$$A \to A - \{A, P\cdot\xi\}^{**}(\mathrm{i}P^2)^{-1}P\cdot\xi \tag{3.24}$$

and continue using the Dirac brackets $\{\ ,\ \}^{**}$. We find

$$\begin{aligned} \xi^\mu(\tau) &\to (g^{\mu\nu} - P^\mu P^\nu/P^2)\xi_\nu(\tau) \equiv \Xi^\mu \\ x^\mu(\tau) &\to x^\mu(\tau) - \mathrm{i}\xi^\mu(\tau)P\cdot\xi(\tau)/P^2 \equiv R^\mu(\tau) \\ \mathrm{i}\xi^\mu(\tau)\xi^\nu(\tau) &\to \mathrm{i}\Xi^\mu\Xi^\nu \equiv W^{\mu\nu} \end{aligned} \tag{3.25}$$

and the generators become

$$\begin{aligned} M^{\mu\nu}(\tau) &= P^\mu R^\nu(\tau) - P^\nu R^\mu(\tau) + W^{\mu\nu} \\ D(\tau) &= P\cdot R(\tau) \\ G^\mu(\tau) &= -\mathrm{i}\Xi^\mu + \mathrm{i}P^\mu\xi_5/mc \end{aligned} \tag{3.26}$$

where we see that $-\mathrm{i}\Xi^\mu$ generates supersymmetry transformations transverse to P^μ, and $\mathrm{i}P^\mu\xi_5/mc$ generates supersymmetry transformations along P^μ. We note that $R^\mu(\tau)$, P^μ, $W^{\mu\nu}$, and M $(=m)$ satisfy the algebra of the Weyl group $\otimes$ U(1), equations

(2.4), in terms of Dirac brackets:

$$\{R^\mu(\tau), P^\nu\}^{**} = -g^{\mu\nu}, \qquad \{P^\mu, P^\nu\}^{**} = 0, \qquad \{P^\mu, W^{\rho\sigma}\}^{**} = 0$$

$$\{R^\mu(\tau), R^\nu(\tau)\}^{**} = -\frac{W^{\mu\nu}}{P^2}, \qquad \{R^\mu(\tau), W^{\rho\sigma}\}^{**} = -\frac{(P^\rho W^{\sigma\mu} - P^\sigma W^{\rho\mu})}{P^2}$$

$$\begin{aligned}\{W^{\mu\nu}, W^{\rho\sigma}\}^{**} &= W^{\mu\rho}\left(g^{\nu\sigma} - \frac{P^\nu P^\sigma}{P^2}\right) - W^{\mu\sigma}\left(g^{\nu\rho} - \frac{P^\nu P^\rho}{P^2}\right) \\ &\quad + W^{\nu\sigma}\left(g^{\mu\rho} - \frac{P^\mu P^\rho}{P^2}\right) - W^{\nu\rho}\left(g^{\mu\sigma} - \frac{P^\mu P^\sigma}{P^2}\right)\end{aligned} \tag{3.27}$$

$$\{P^\mu, M\}^{**} = \{R^\mu(\tau), M\}^{**} = \{W^{\mu\nu}, M\}^{**} = 0.$$

(The Dirac brackets of the generators $M^{\mu\nu}$ and D can easily be constructed from equations (3.26) and (3.27).) We also note their Dirac brackets with Ξ^μ

$$\begin{aligned}&\{R^\mu(\tau), \Xi^\nu\}^{**} = \frac{P^\nu \Xi^\mu}{P^2}, \qquad \{P^\mu, \Xi^\nu\}^{**} = 0, \qquad \{M, \Xi^\nu\}^{**} = 0 \\ &\{W^{\mu\nu}, \Xi^\sigma\}^{**} = -\Xi^\mu\left(g^{\nu\sigma} - \frac{P^\nu P^\sigma}{P^2}\right) + \Xi^\nu\left(g^{\mu\sigma} - \frac{P^\mu P^\sigma}{P^2}\right)\end{aligned} \tag{3.28}$$

which should be compared with equations (2.8a). The Dirac bracket of Ξ^μ with itself is

$$\{\Xi^\mu, \Xi^\nu\}^{**} = \mathrm{i}\left(g^{\mu\nu} - \frac{P^\mu P^\nu}{P^2}\right). \tag{3.29}$$

Equations (3.27) are invariant under the supersymmetry transformation generated by $-\mathrm{i}\Xi^\mu (= G^\mu - P^\mu P \cdot G/P^2)$:

$$\begin{aligned}&R^\mu(\tau) \to R^\mu(\tau) + \frac{\mathrm{i}\alpha \cdot P \Xi^\mu}{P^2} \\ &W^{\mu\nu} \to W^{\mu\nu} + \mathrm{i}\left(\left(\alpha^\mu - \frac{P^\mu P \cdot \alpha}{P^2}\right)\Xi^\nu + \Xi^\mu\left(\alpha^\nu - \frac{P^\nu P \cdot \alpha}{P^2}\right)\right) \\ &[M^{\mu\nu} \to M^{\mu\nu} + \mathrm{i}(\alpha^\mu \Xi^\nu + \Xi^\mu \alpha^\nu)]\end{aligned} \tag{3.30}$$

where α^μ is an infinitesimal odd Grassmann parameter (cf equations (2.9a)). As in the nonrelativistic case, an infinitesimal rest-frame rotation generated by $\varepsilon_{\mu\nu} W^{\mu\nu}$ ($\varepsilon_{\mu\nu}$ an infinitesimal antisymmetric c-number parameter) can be considered as a supersymmetry transformation with a Ξ^μ-dependent parameter since $\varepsilon_{\mu\nu} W^{\mu\nu} = \mathrm{i}(\varepsilon_{\mu\nu}\Xi^\mu)\Xi^\nu$. We also note the Dirac brackets with $\mathrm{i}P^\mu \xi_5/mc$ evaluated in the rest-frame (i.e. $(P^2)^{1/2}\xi_5/mc$):

$$\{R^\mu(\tau), \mathrm{i}(P^2)^{1/2}\xi_5/mc\}^{**} = -\mathrm{i}P^\mu \xi_5/(P^2)^{1/2}mc, \tag{3.31}$$

all others zero.

We now apply the mass-shell constraint $\chi \equiv P^2 - m^2c^2 \approx 0$, together with the gauge-fixing constraint $D(\tau) - \gamma - (P^2)^{1/2}c\tau \approx 0$ which says that $R^0(\tau)$ evaluated in the rest-frame (i.e. $L^{-1}(P)^0_\mu R^\mu(\tau)$) is equal to $(\gamma/mc) + c\tau$ (in other words, we choose the evolution function of equations (2.7) $f(P^2, \tau) = (P^2)^{1/2}c\tau$). These form a set of second-class constraints so we need the new Dirac brackets

$$\{A, B\}^{***} = \{A, B\}^{**} - \{A, \chi_\alpha\}^{**} C^{-1}_{\alpha\beta} \{\chi_\beta, B\}^{**} \tag{3.32}$$

where $\chi_1 \equiv \chi$, $\chi_2 \equiv D(\tau) - \gamma - (P^2)^{1/2} c\tau$, and

$$C_{\alpha\beta} = \{\chi_\alpha, \chi_\beta\} = 2P^2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.33}$$

The Weyl RPW algebra, equations (3.27), becomes the Poincaré RPW algebra (cf Rohrlich 1977, Almond 1978):

$$\begin{aligned}
&\{R^\mu(\tau), P^\nu\}^{***} = -\left(g^{\mu\nu} - \frac{P^\mu P^\nu}{m^2c^2}\right), \qquad \{P^\mu, P^\nu\}^{***} = 0, \qquad \{P^\mu, W^{\rho\sigma}\}^{***} = 0 \\
&\{R^\mu(\tau), R^\nu(\tau)\}^{***} = -\frac{M^{\mu\nu}}{m^2c^2}, \qquad \{R^\mu(\tau), W^{\rho\sigma}\}^{***} = -\frac{(P^\rho W^{\sigma\mu} - P^\sigma W^{\rho\mu})}{m^2c^2} \\
&\{W^{\mu\nu}, W^{\rho\sigma}\}^{***} = W^{\mu\rho}\left(g^{\nu\sigma} - \frac{P^\nu P^\sigma}{m^2c^2}\right) - W^{\mu\sigma}\left(g^{\nu\rho} - \frac{P^\nu P^\rho}{m^2c^2}\right) \\
&\qquad + W^{\nu\sigma}\left(g^{\mu\rho} - \frac{P^\mu P^\rho}{m^2c^2}\right) - W^{\nu\rho}\left(g^{\mu\sigma} - \frac{P^\mu P^\sigma}{m^2c^2}\right) \\
&\{P^\mu, M\}^{***} = \{R^\mu(\tau), M\}^{***} = \{W^{\mu\nu}, M\}^{***} = 0.
\end{aligned} \tag{3.34}$$

Since $\{P^2 - m^2c^2, \Xi^\mu\}^{***} = 0 = \{D(\tau) - \gamma - (P^2)^{1/2} c\tau, \Xi^\mu\}^{***}$, equations (3.28) and equations (3.29) are unchanged except for $P^2 \to m^2c^2$ on the RHS. Equation (3.31) becomes

$$\left\{R^\mu(\tau), \frac{\mathrm{i}(P^2)^{1/2}\xi_5}{mc}\right\}^{***} = 0. \tag{3.35}$$

The simplicity of these results is because $M = m$ rather than a function of Ξ^μ and ξ_5, when the Dirac brackets $\{\ ,\ \}^{***}$ would have been much more complicated, as in the case of the string (Almond 1978).

4. Quantisation

We now wish to construct a representation for the operators R^μ, P^μ, $W^{\mu\nu}$, M, Ξ^μ, and ξ_5 which satisfies equations (3.27) and (3.28) with $\{A, B\}^{**} \to [A, B]/\mathrm{i}\hbar$, and the quantum versions of equations (3.29) and (3.21), which are the Clifford algebra C_d:

$$\begin{aligned}
&[\Xi^\mu, \Xi^\nu]_+ = -\hbar\left(g^{\mu\nu} - \frac{P^\mu P^\nu}{P^2}\right) \\
&[\Xi^\mu, \xi_5]_+ = 0, \qquad [\xi_5, \xi_5]_+ = -\hbar.
\end{aligned} \tag{4.1}$$

Defining the matrix operator

$$L^{-1}(P)^\mu_{\ \nu} = \begin{pmatrix} P^0/(P^2)^{1/2} & P_j/(P^2)^{1/2} \\ -P^i/(P^2)^{1/2} & \delta^i_j - P^iP_j/(P^2)^{1/2}((P^2)^{1/2} + P^0) \end{pmatrix} \tag{4.2}$$

which takes P^ν into $((P^2)^{1/2}, \mathbf{0})$, we can write equations (4.1) as

$$[\xi^i(P), \xi^j(P)]_+ = \hbar\delta^{ij}, \qquad [\xi^i(P), \xi_5]_+ = 0, \qquad [\xi_5, \xi_5]_+ = -\hbar \tag{4.3}$$

where $\xi^i(P) = L^{-1}(P)^i_{\ \nu}\Xi^\nu$. We can construct a faithful matrix representation γ^μ of the Clifford algebra C_d by standard methods (for a good mini-review of Clifford algebras and their representations see the Appendix of Casalbuoni (1976a)), and a representation of equations (4.3) is then given by

$$\left.\begin{aligned}\xi^i(P)&\to(\tfrac{1}{2}\hbar)^{1/2}\gamma^i\gamma_5\\ \xi_5&\to-(\tfrac{1}{2}\hbar)^{1/2}\mathrm{i}\gamma_5\end{aligned}\right\}\quad \text{for } d=3,4,7,8,11,12,\ldots \tag{4.4a}$$

$$\left.\begin{aligned}\xi^i(P)&\to(\tfrac{1}{2}\hbar)^{1/2}\gamma^i(\mathrm{i}\gamma_5)\\ \xi_5&\to(\tfrac{1}{2}\hbar)^{1/2}\gamma_5\end{aligned}\right\}\quad \text{for } d=1,2,5,6,9,10,\ldots \tag{4.4b}$$

where $\gamma_5=\mathrm{i}\gamma^0\gamma^1\ldots\gamma^{d-1}$ satisfies $(\gamma_5)^2=+1$ for $d=3,4,7,\ldots$ and $(\gamma_5)^2=-1$ for $d=1,2,5,\ldots$. The operator Ξ^μ is then given by $\Xi^\mu=L(P)^\mu_i\xi^i(P)$ so its representation is given by

$$\begin{aligned}\Xi^0&\to\frac{\boldsymbol{p}\cdot((\tfrac{1}{2}\hbar)^{1/2}\boldsymbol{\gamma}\gamma_5)}{(p^2)^{1/2}}\\ \Xi^i&\to(\tfrac{1}{2}\hbar)^{1/2}\left(\gamma^i\gamma_5+\frac{p^i\boldsymbol{p}\cdot\boldsymbol{\gamma}\gamma_5}{[(p^2)^{1/2}((p^2)^{1/2}+p^0)]}\right)\end{aligned}\tag{4.5}$$

for $d=3,4,7,\ldots$, with the same expressions multiplied by i for $d=1,2,5,\ldots$. The spin operator $S^{ij}(P,W)$ is given by

$$\begin{aligned}S^{ij}(P,W)&=-L^{-1}(P)^i_\mu L^{-1}(P)^j_\nu W^{\mu\nu}=-L^{-1}(P)^i_\mu L^{-1}(P)^j_\nu\tfrac{1}{2}\mathrm{i}[\Xi^\mu,\Xi^\nu]\\ &=-\tfrac{1}{2}\mathrm{i}[\xi^i(P),\xi^j(P)]\end{aligned}\tag{4.6}$$

so its representation is

$$S^{ij}(P,W)\to\tfrac{1}{2}\hbar\sigma^{ij}\tag{4.7}$$

where $\sigma^{ij}=\tfrac{1}{2}\mathrm{i}[\gamma^i,\gamma^j]$. We must now construct a representation for R^μ and $W^{\mu\nu}$. Fortunately, this has already been done for us for $d=4$ (Almond 1973, equations (3.29) and (3.38)), and the generalisation to arbitrary d is straightforward:

$$\begin{aligned}W^{0i}&\to-\tfrac{1}{2}\hbar p^j\sigma^{ji}/(p^2)^{1/2}\\ W^{ij}&\to-\tfrac{1}{2}\hbar\left(\sigma^{ij}+\frac{p^ip^k\sigma^{kj}-p^jp^k\sigma^{ki}}{(p^2)^{1/2}((p^2)^{1/2}+p^0)}\right)\\ R^0&\to-\mathrm{i}\hbar\left(\frac{\partial}{\partial p^0}\right)_{\boldsymbol{p}}\\ R^i&\to\mathrm{i}\hbar\left(\frac{\partial}{\partial p^i}\right)_{p^0}+\frac{\tfrac{1}{2}\hbar p^j\sigma^{ji}}{(p^2)^{1/2}((p^2)^{1/2}+p^0)}.\end{aligned}\tag{4.8}$$

The verification that equations (4.4), (4.5), and (4.8) (together with $P^\mu\to p^\mu$, $M\to m$) satisfy the quantum version of equations (3.27) and (3.28) is a matter of straightforward (though somewhat tedious) algebra.

An on-mass-shell particle is described by the above representation with the quantum states of the system $|\Phi\rangle$ restricted to those satisfying $\langle p|\Phi\rangle=\delta(p^2-m^2c^2)\Phi(\boldsymbol{p})/(\delta(0))^{1/2}$, and with the gauge-fixing constraint $\chi_2\equiv D(\tau)-\gamma-(P^2)^{1/2}c\tau\approx0$ applied to such states as a matrix element condition. The discussion is far from trivial and is given in detail elsewhere (Almond 1980).

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